GEOMETRIC CHARACTERISTICS OF CONICS IN BÉZIER FORM

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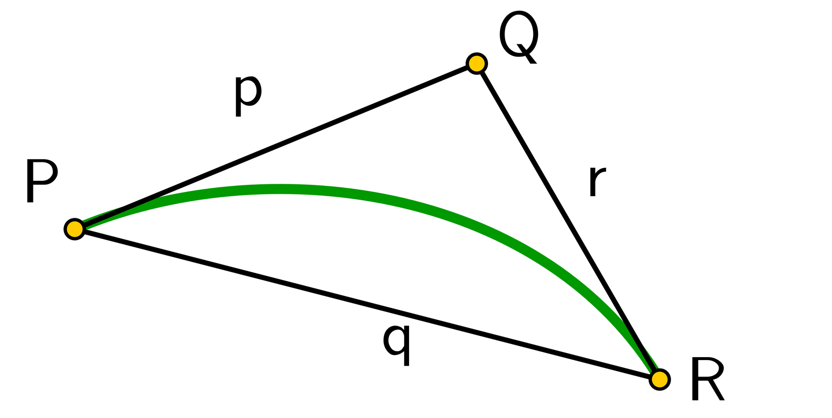
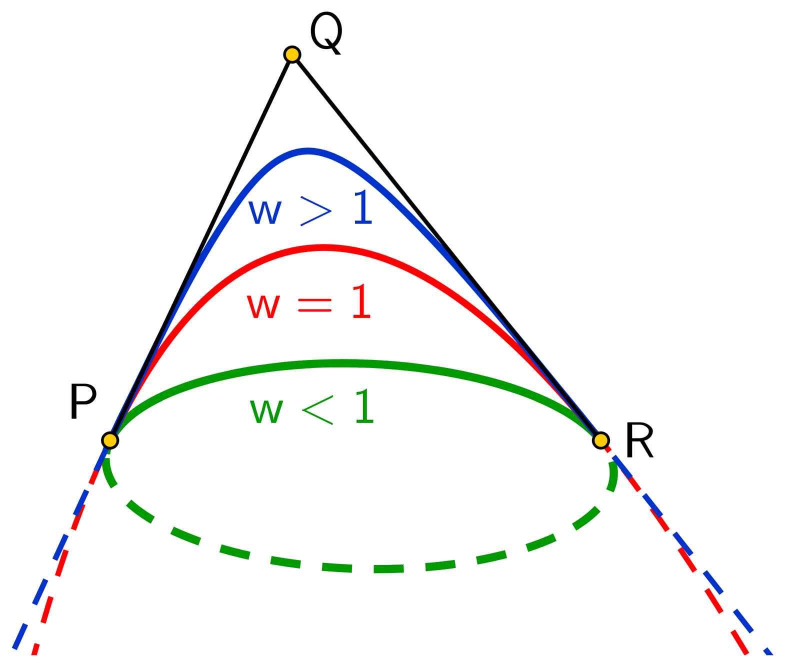
Abstract

Synthetic derivation of closed formulae of the geometric characteristic of a conic given in Bézier form in terms of its control polygon, {P, Q, R}, and weights, {1, w, 1}.



Conics in Bézier form



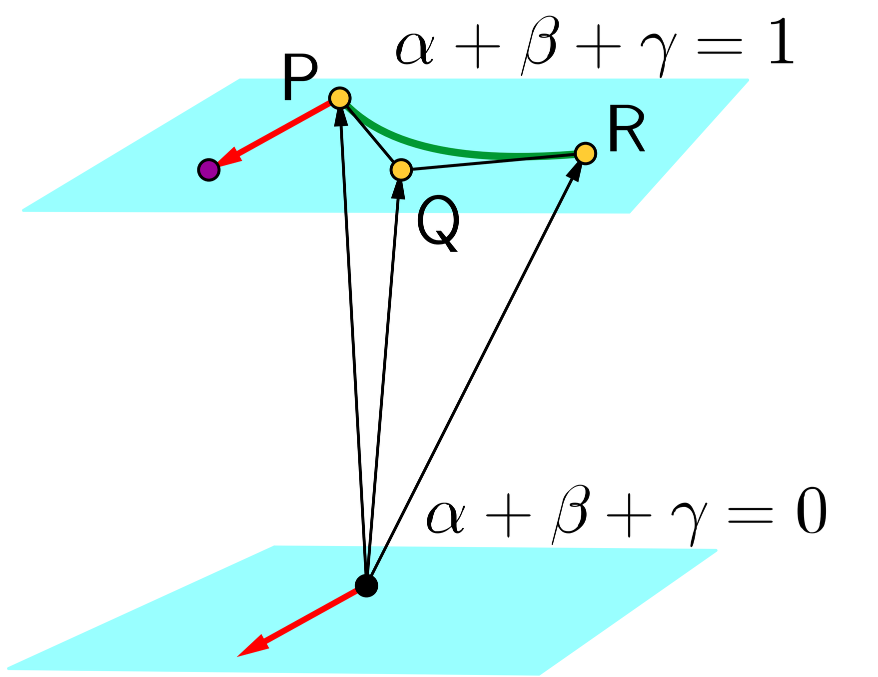
In the Bézier formalism, an arc of a conic is a rational curve of degree 2 with control polygon {P, Q, R} for which the weights can be normalized to {1, w, 1}. The parametrization of the conic arc is

$$C(t)=\frac{(1-t)^2\mathrm{P}+2\mathrm{w}t(1-t)\mathrm{Q}+t^2\mathrm{R}}{(1-t)^2+2\mathrm{w}t(1-t)+t^2},\qquad t\in[0,1].$$

The weight, w, classifies the type of conic:

- $\mathrm{w}<1$, ellipse,
- $\mathrm{w}=1$, parabola,
- $\mathrm{w}>1$, hyperbola.

Notation



Consider the frame, {P, Q, R}, defined by the points of the control polygon of the conic. Let $(\alpha,\beta,\gamma)\in\mathbb{P}^2$ be the coordinates of a point in this frame.

The conic lies on the **affine plane** defined by points whose coordinates satisfy $\alpha+\beta+\gamma=1$.

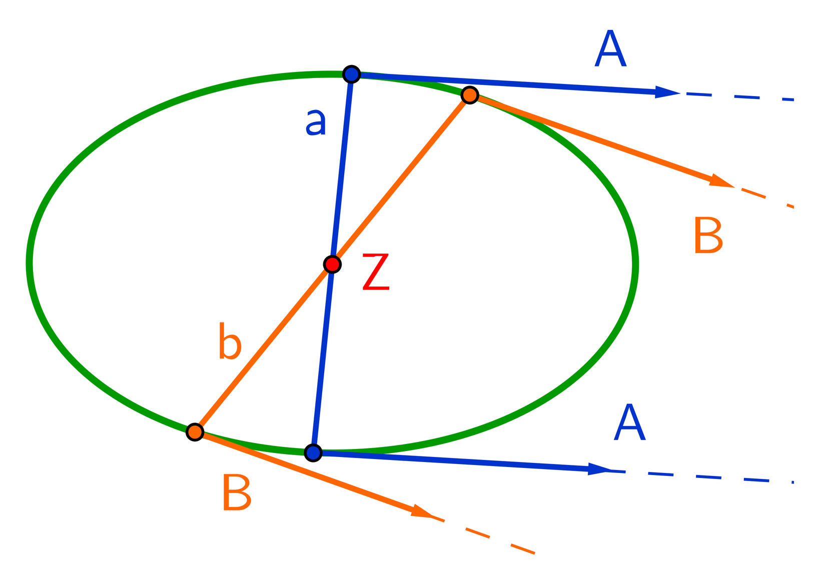
Points on the **line at infinity**, z, satisfy $\alpha+\beta+\gamma=0$. A point in z is a direction on the affine plane.

The dual frame, $\{p,q,r\}$, of linear forms is associated to the polar lines of {P, Q, R}, normalized so that $1=p(\mathrm{R})=q(\mathrm{Q})=r(\mathrm{P})$. In this frame, lines on the affine plane have coordinates (π,ρ,σ).

Affine invariants

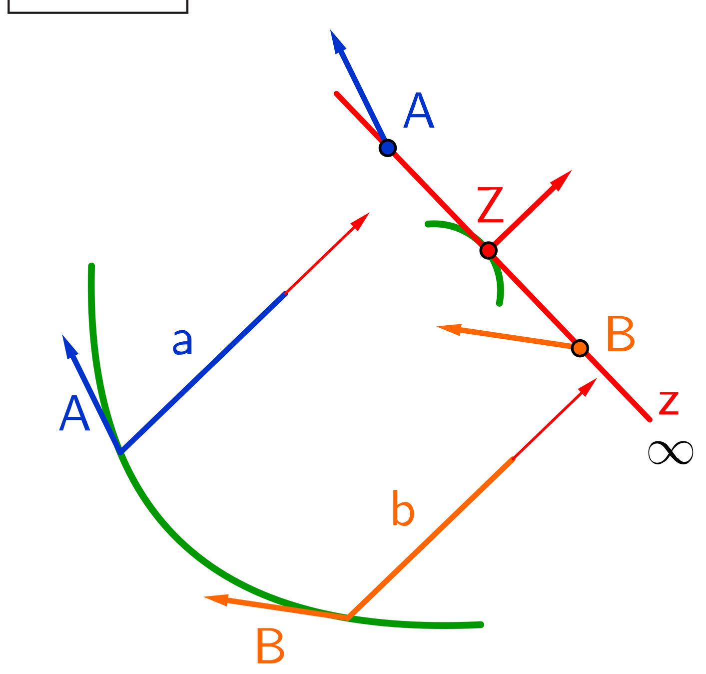
The **center** of the conic, Z, has the line at infinity, z, as its polar line.

$\mathbf{w}\neq 1$



$$\mathrm{Z}=\frac{\mathrm{P}+\mathrm{R}-2\mathrm{w}^2\mathrm{Q}}{2-2\mathrm{w}^2}$$

$\mathbf{w}=1$



$$\mathrm{Z}=\frac{\mathrm{P}+\mathrm{R}}{2}-\mathrm{Q}$$

Diameters are polar lines of points on z.

Asymptotes are polar lines of points on the conic that lie at infinity. A hyperbola has two asymptotes, a parabola one and an ellipse none.

The asymptotes of a hyperbola have equations

$$\left(\mathrm{w}\pm\sqrt{\mathrm{w}^2-1}\right)p(X)+\left(\mathrm{w}\mp\sqrt{\mathrm{w}^2-1}\right)r(X)+\frac{q(X)}{\mathrm{w}}=0$$

The equation for the double asymptote of the parabola ($\mathrm{w}=1$)

$$p(X)+r(X)+q(X)=0$$

Implicit equations

Consider the conic with control polygon {P, Q, R}, and weights, {1, w, 1}.

The implicit equation of the (point) conic is

$$p(X)r(X)-\frac{q(X)^2}{4\mathrm{w}^2}=0,\ \text{or, in coordinates,}\ \ \alpha\gamma-\frac{\beta^2}{4\mathrm{w}^2}=0.$$

The implicit tangential equation of the conic

$$x(P)x(R)-\mathrm{w}^2x(Q)^2=0,\ \text{or, in coordinates,}\ \ \pi\sigma-\mathrm{w}^2\rho=0.$$

Euclidean invariants

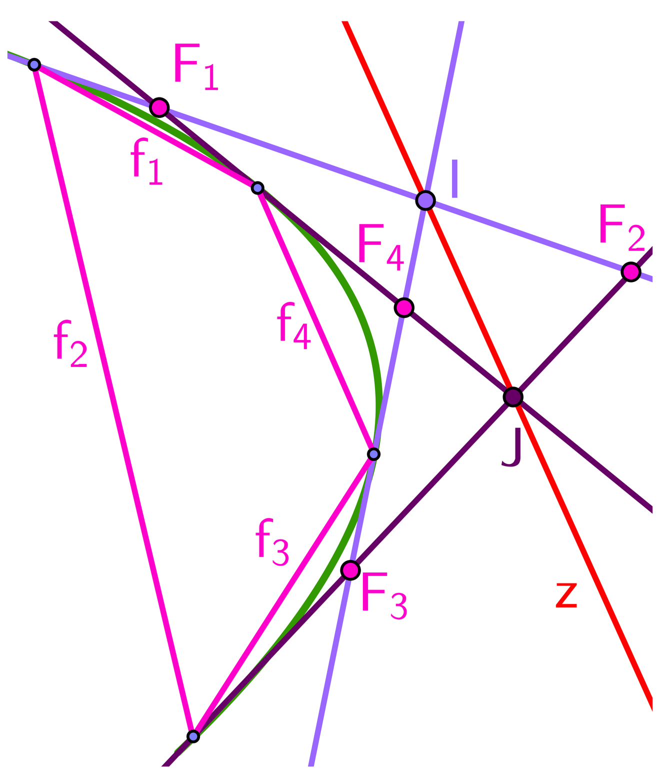
Circular or **isotropic points** are points (with complex coordinates) on the line at infinity, z, through which all circles in the Euclidean plane pass.

The **foci** are the points of intersection of the two pairs of tangent lines to the conic from the circular points, I, J:

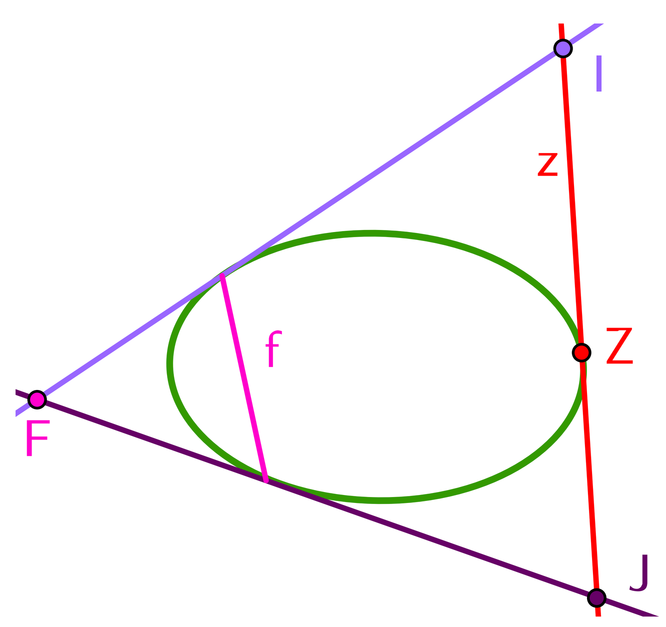
- Central conics have four foci, two real and two complex.
- Parabolas only have one focus, since they are tangent to z at Z.

The **directrices** are the polar lines of the foci, two real and two complex, except for the parabola that has only one.

$\mathbf{w}\neq 1$



$\mathbf{w}=1$



$$\mathrm{F}=\frac{\|\vec r\|^2\mathrm{P}+2\langle\vec p,\vec r\rangle\mathrm{Q}+\|\vec p\|^2\mathrm{R}}{\|\vec p+\vec r\|^2}$$

with $\vec p=\mathrm{P}-\mathrm{Q}$, $\vec r=\mathrm{R}-\mathrm{Q}$

An **axis** is a diameter orthogonal to the tangent lines to the conic at their intersection points.

If $\mathrm{w}\neq 1$, $\quad(\mathrm{w}^2\rho+1)p(X)+\rho q(X)+(\mathrm{w}^2\rho-1)r(X)=0$

with ρ a solution of $\mathrm{w}^2(\mathrm{w}^2-1)\rho^2+\frac{2\mathrm{w}^2\left(\|\vec p\|^2+\|\vec r\|^2\right)-\|\vec q\|^2}{\|\vec p\|^2-\|\vec r\|^2}\rho+1=0.$

If $\mathrm{w}=1$, $\quad 2\langle\vec r,\vec p+\vec r\rangle p(X)+\left(\|\vec r\|^2-\|\vec p\|^2\right)q(X)-2\langle\vec p,\vec p+\vec r\rangle r(X)=0.$

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